Talk 3. ∞-categories: Introduction.

· Why do we need them?

1. The ~~very~~ objects of derived geometry, e.g. derived schemes, derived affine schemes, prestacks, quasi-coherent sheaves, etc., all form ∞-categories.
2. The very definition of certain objects ~~needs~~ forces us to deal w/ homotopy coherence questions: what are $\mathbb{E}_\infty$-rings? What is the coherent notion of a ringed spaces where the rings are derived rings, i.e. correct notion of an ∞-topos,? Stacks are sheaves of spaces, but treated up to homotopy.
3. Descent statements. We can describe the ordinary category of quasi-coherent sheaves on $\mathbb{P}^1$ by sheaves on each of the basic opens U_0, U_1 + a glueing data, i.e. isomorphism between the restrictions to $U_0 \cap U_1$.

This fails for the triangulated category $D(\mathbb{P}^1)$, i.e. the derived (ordinary) category of quasi-coherent sheaves: $D(\mathbb{P}^1) \neq \lim_{\Delta^{op}} D((\mathcal{U}/\mathbb{P}^1)^\bullet)$
$\mathcal{U} \to \mathbb{P}^1$ a Zariski cover.

[$\mathrm{RHom}_{D(\mathbb{P}^1)}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(-2)[1]) \neq 0$ however $\mathcal{O}_{\mathbb{P}^1}|_{U_0/U_1}$ is $k[t]$ in degree 0 or -1 & $k[t,t^{-1}] \to k[t,t^{-1}][1]$ is zero.].

However, if one consider the derived ∞-category $\mathrm{QCoh}(\mathbb{P}^1)$ one recovers: $\mathrm{QCoh}(\mathbb{P}^1) \simeq \mathrm{QCoh}(U_0) \times_{\mathrm{QCoh}(U_{01})} \mathrm{QCoh}(U_1)$. $\leftarrow$ homotopy fiber product.

4. Treat moduli spaces w/ automorphisms, picture from last time:

$\mathcal{X}: \mathrm{CAlg} \to \mathrm{Spc}$. $\leftarrow$ needs to be treated as an ∞-cat.

Basic definitions.

Def'n: An ∞-category is a simplicial set $S_\bullet : \Delta^{op} \to Sets$ satisfying the following property: (weak Kan condition).

$\forall\, n \geq 2$, and $0 < i < n$ a ~~dotted~~ dashed arrow exists.

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & S_\bullet \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \quad (\star)$$

$\Lambda^n_i :=$ delete the $(n-1)$-face opposite to the vertex i. (i^{th} horn).

e.g.

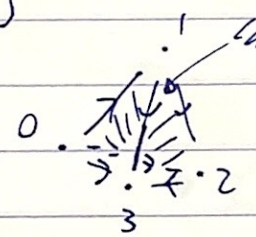
$$\Lambda^2_0 = \begin{tikzcd} 0 \arrow[r] \arrow[dr] & 1 \\ & 2 \end{tikzcd}$$

$$\Lambda^2_1 = \begin{tikzcd} 0 \arrow[r] & 1 \arrow[d] \\ & 2 \end{tikzcd}$$

$\Delta^n_m = \mathrm{Hom}_\Delta([m],[n])$ $(\star)$ = non-decreasing fcts.

$(\Lambda^n_i)_m := \{\alpha \in \Delta^n_m \mid [n] \not\subseteq \alpha([m]) \cup \{i\}\}$.

$\Lambda^3_1 =$



back painted

interior empty and no face on

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 2 \\ 3 & \end{tikzcd}$$

Examples: (i) any Kan complex, i.e. $K_\bullet : \Delta^{op} \to Sets$ s.t. $(\star)$ is satisfied for $0 \leq i \leq n$.

X top. space. $\mathrm{Sing}_n(X) := \mathrm{Hom}_{Top}(|\Delta^n|, X)$ is a Kan complex.

(ii) let C be an (ordinary) category,

$$N_n C := \mathrm{Fun}([n], C).$$

$N_0 C \simeq \mathrm{Ob}(C)$

$N_1 C = \{f : X \to Y \mid X, Y \in \mathrm{Ob}(C)\}$.

$N_2 C = \{X \xrightarrow{f} Y \xrightarrow{g} Z\}$.

$N_\bullet C$ satisfy $(\star)$ with an unique. ~~dotted~~ dashed arrow.

(this is actually all examples of $(\star)$ w/ unique lift.)

(iii) products and coproducts of ∞-categories. (as simplicial sets) i.e. $f:\Delta^1 \to \mathcal{C}$.

Def'n: Given $f, g : X \to Y$ morphism in an ∞-cat. $\mathcal{C}$ a homotopy between f and g is the data of $\sigma : \Delta^2 \to \mathcal{C}$ s.t. the boundary of σ is

$$\begin{tikzcd} & Y \arrow[dr, "id_Y"] & \\ X \arrow[ur, "f"] \arrow[rr, "g"'] & & Y \end{tikzcd}.$$

Exercise: (i) The notion of homotopy defines an equivalence relation on the set of morphisms between two objects X and Y. $(f \sim g)$.
(ii) If $\mathcal{C} = N.C$ then $f \sim g \Leftrightarrow f = g$.

Warning: Given $X \xrightarrow{f} Y \xrightarrow{g} Z$ in an ∞-cat. $\mathcal{C}$,
(i.e. $\Lambda^2_1 \to \mathcal{C}$.), $\exists$ $h : X \to Z$ s.t. one has:

$$\begin{tikzcd} & Y \arrow[dr, "g"] \arrow[d, phantom, "\sigma"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & {} & Z \end{tikzcd}, \text{ i.e. } \sigma : \Delta^2 \to \mathcal{C}.$$

h is a composite of g & f, and σ is a witness of their composition, i.e. it is imposing the equation "$h = g \circ f$". Neither h nor σ is unique!

Rk: The homotopy class of h is unique. (Exercise easy.)
The space of σ's that witness a composition is contractible (Exercise harder.)

Def'n: Given X, Y objects in $\mathcal{C}$ their mapping space is the simplicial set:

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) := \{X\} \times_{\mathrm{Fun}(\{0\},\mathcal{C})} \mathrm{Fun}(\Delta^1,\mathcal{C}) \times_{\mathrm{Fun}(\{1\},\mathcal{C})} \{Y\}.$$

Given $K \in Sets_\Delta$

Rk: - $\mathrm{Fun}(K, \mathcal{C}) :=$ simplicial set of morphisms of simplicial sets,
i.e. $\mathrm{Fun}(K,\mathcal{C})_n := \mathrm{Hom}_{Sets_\Delta}(K \times \Delta^n, \mathcal{C})$.

- $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is a Kan complex. (check it!)
- Some references use the notation: $\mathrm{Maps}_{\mathcal{C}}(X,Y)$ and use $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ for $\mathrm{Hom}_{h\mathcal{C}}(X,Y)$ (see below).
- There are many more models for $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ which are homotopy equivalent to it. (see §1.2 HTT).

As we mentioned last week the object $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is only meaningful as a homotopy type, i.e. as an object of the (ordinary) ~~category~~ homotopy category of spaces. $\underline{\mathcal{T}op}$.

Recall we can construct $\underline{\mathcal{T}op}$ as follows:

- Consider $\mathrm{Kan} \subset Sets_\Delta$ the ordinary category of Kan complexes.
- given $g, f : X \to Y$ a maps of ~~a~~ Kan complexes we say $g \sim f$ if they are homotopic in the sense above.
  - this gives a (homotopy) equivalence relation.
  - $\underline{\mathcal{T}op} :=$ Same objects as Kan
    morphisms are homotopy equivalence classes. (Check it is well-defined for composition.)

Prop: Given X, Y, Z objects in $\mathcal{C}$ one has a morphism:
$$\circ : \mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}}(X,Z) \quad \text{in } \underline{\mathcal{T}op}\,!$$

Pf: $\mathrm{Hom}_{\mathcal{C}}(X,Y,Z) := \mathrm{Fun}(\Delta^2, \mathcal{C}) \times \{X,Y,Z\}$.
$\mathrm{Fun}(\{0,1,2\},\mathcal{C})$.

$$\begin{array}{c} \mathrm{Hom}_{\mathcal{C}}(X,Y,Z) \\ \Theta \downarrow \\ \mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y). \end{array}$$

$\partial \Delta^n_m$
$\shortparallel$
$\{f \in \Delta^n_m \mid f$ is not surjective$\}$.

Claim: Θ is a trivial Kan fibration.
i.e. R.L.P. w.r.t. $\partial\Delta^n \hookrightarrow \Delta^n$.
$\Rightarrow [\Theta]$ is an isom. on $\underline{\mathcal{T}op}$.

$[\Theta]^{-1}$ restrict to $\Delta^1 \hookrightarrow \Delta^2$ $(0 \to 2)$

Then $\mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}}(X,Y,Z) \to \mathrm{Hom}_{\mathcal{C}}(X,Z)$. ☒

<u>Def'n:</u> Given two ∞-cats $\mathcal{C}$ and $\mathcal{D}$ a <u>functor</u> $F : \mathcal{C} \to \mathcal{D}$ is a map (i.e. morphism in the (ord.) category Sets_Δ) of simplicial sets. (One could be tempted to write ∞-functor for those, but no one does that!)

later, after examples of functors.

<u>Prop:</u> $\mathcal{C}$ an ∞-cat. & K a simplicial set, then

$$\mathrm{Fun}(K, \mathcal{C}) \text{ is an } \infty\text{-category.}$$

The proof is not hard but needs some combinatorics of simplicial sets. See Kerodon Thm 1.4.3.7.

<u>Rk:</u> - Notice that the data of a functor $F : \mathcal{C} \to \mathcal{D}$ is a lot of data, in particular takes witnesses to witnesses.

- <u>Prop.</u> above is a very good formal property to have, other models lack this. (see below).
- See [HTT, §1.2.6] for a discussion of what the above def'n means for commutative diagrams. (also Kerodon §1.4.2).

<u>Def'n:</u> Given $\mathcal{C}$ an ∞-cat. let $h\mathcal{C}$ denote the ordinary category w/:

- objects same as $\mathcal{C}$.
- morphisms are equivalence class (w.r.t. $\sim$) of morphisms in $\mathcal{C}$.

This is called the <u>homotopy category</u> of $\mathcal{C}$.

Kerodon §1.3.5.

<u>Exercises:</u> (i). The construction above is well-defined (composition works!)
(ii) $\exists$ a map $\mathcal{C} \to N h. \mathcal{C}$ s.t. $\mathrm{Hom}_{\mathrm{Cat}}(h\mathcal{C}, D) \to \mathrm{Hom}_{\mathrm{Sets}_\Delta}(\mathcal{C}, N.(D))$ is an equivalence.